NP-completeness Continued...

To really understand NP-completeness, you will need to read Chapter 36. This handout summarizes the basic concepts we discussed in the first class on NP-completeness.

We briefly review the reductions done in class that are not in the textbook.

0.1 Graph Coloring

Graph 3-coloring Problem: Given a graph $G = (V, E)$ is there a way to color the vertices of G with 3 colors? A valid coloring of the graph requires all adjacent nodes to have different colors.

Graph 4-coloring Problem: Given a graph $G' = (V', E')$ is there a way to color the vertices of G' with 4 colors? A valid coloring of the graph requires all adjacent nodes to have different colors.

There is a reduction that reduces 3-SAT to Graph 3-Coloring; thus showing that Graph 3-Coloring is NP-hard. Since it is easy to verify a proposed 3 coloring in polynomial time, the problem is also in NP, and thus NP-Complete.

We now show how to reduce 3-Coloring to 4-Coloring. Given a graph G, we create G' as follows. Add a new vertex v to G and add an edge from v to each vertex of V. $V' = V \cup \{v\}$. $E' = E \cup \{(v, u) | u \in V\}$.

To prove the correctness of the reduction observe that if G is 3 colorable then G' is 4 colorable (use the new color for v). If G' is 4 colorable, then assuming that v uses one color, the remaining vertices of G' use only 3 colors and this shows that G is 3 colorable.

0.2 Dominating Set

Given a graph $G = (V, E)$ and an integer K, does G contain a dominating set of size at most K? (A dominating set is a subset of vertices S with the property that each vertex in G is either in S or has a neighbor in S.)

We will show that this problem is NP-hard by reducing the known NP-complete problem Vertex Cover to it.

Vertex Cover: given a graph $G' = (V', E')$ and an integer K' does G' contain a vertex cover of size at most K'? (A vertex cover is a subset of vertices S' with the property that each edge has at least one endpoint in S'.)

For the reduction, we create G as follows. For each edge (x, y) in G' add a new vertex xy and add edges (xy, x) and (xy, y). (In other words we have replaced each edge by a triangle.) Set $K = K'$. We also assume that the input graph has no vertices of zero degree (for example this is true for the reduction we did from CLIQUE to VERTEX COVER.)

Claim: There exists a vertex cover in G' of size K' if and only there exists a dominating set in G of size K'.

Proof: If S' is a vertex cover in G' then S' is also a dominating set in G: for a node of G that is also in G', if the node has at least one edge incident on it, then the other end of the edge is in S' if the node itself is not in S'. For a new node of the type xy, we know that either x or y is in S', and so the node has a neighbor in S'. Thus S' is a dominating set in G'.

To prove the converse, note that if S is a dominating set in G, then without loss of generality the vertices in S are also in G'. If a node of the form xy is in the dominating set, then we can remove xy and add x (or y) to S. S is still a dominating set. Now note that S is a vertex cover in G' since for each edge (x, y) in G', since the node xy is dominated in G, either x or y is in S. This completes the proof.